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Generalised XY model

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Abstract. The methods of Lie algebras are used to construct and solve a generalisation of the XY model.

1. Introduction

In this paper we apply the methods of Lie algebras to solve and generalise the XY model (Lieb *et al* 1961, Katsura 1962). The methods employed originate in the spectrum generating algebras of particle physics—which are non-symmetry algebras of the Hamiltonian, and provide elegant solutions for quantum statistical problems (Solomon 1971, 1974). We briefly outline the approach we adopt in the present context.

In equilibrium statistical mechanics the thermodynamic behaviour of a system, whose Hamiltonian is H , follows from evaluation of the partition function

$$Q = \text{Tr}[\exp(-\beta H)],$$

where β is the inverse of the absolute temperature times Boltzmann's constant. Classically, the trace may be interpreted as the sum over all the allowed configurations of the system; in quantum mechanics, as the usual Hilbert space trace.

In our algebraic treatment we shall consider H to be an element of a suitable Lie algebra of rank l . This means that one can find a Cartan basis for the algebra which includes the l mutually commuting elements, h_1, h_2, \dots, h_l . The solution of the problem is obtained by finding an automorphism of the algebra, implemented by U say, such that

$$H \mapsto UHU^{-1} = \sum_{m=1}^l \Lambda_m h_m \quad (1.1)$$

where the Λ_m are known scalars (elements of the underlying field). Since in principle the spectra of the h_m are known, such an automorphism effects diagonalisation and clearly leaves the partition function Q unchanged.

Therefore the strategy to be adopted is in three parts:

- (a) Determine a *suitable* Lie algebra which is to generate the spectrum of H . The Hamiltonian H will be an element of the algebra in some (usually large) representation.

- (b) Choose a small-dimensional, faithful representation in which to implement the automorphism (1.1).
- (c) Now return to the original representation, in which (1.1) remains true and in particular the values of the scalars Λ_m are unchanged, to evaluate the spectrum of H and the partition function.

In the case of the XY model, on a cyclic lattice of N points, the application of the three-part strategy gives:

- (a) The Hamiltonian is an element of a $(2^N \times 2^N)$ -dimensional representation of $SO(2N) \oplus SO(2N)$. This is a rank $2N$ algebra.
- (b) We implement the automorphism (1.1) in the faithful $(4N \times 4N)$ -dimensional representation, determining the values of the $2N$ constants Λ_m .
- (c) We return to the $2^N \times 2^N$ representation to evaluate the partition function.

The reason that the solution of the XY model is so readily obtained, in spite of the seemingly cumbersome nature of the machinery outlined above (nobody can diagonalize even a $4N \times 4N$ matrix in general!) is that the *translation invariance* of physically interesting models means that in these cases the underlying algebra is a much smaller one, and effectively reduces the computation in all such cases to the diagonalization of a small (in our case, 2×2) matrix. The generalized XY model we treat is in fact the most general translationally invariant model consistent with the $SO(2N) \oplus SO(2N)$ algebra of the original XY model.

Since the automorphism (1.1) reduces the computation of the partition function in (c) to that of a system of uncoupled spins, to which the model is therefore equivalent in an algebraic sense, we now briefly treat such a free spin model.

2. Free spin model

Consider the following Hamiltonian, representing a system of N uncoupled spins:

$$H = - \sum_{m=1}^N \Lambda_m Z_m. \quad (2.1)$$

The Λ_m are positive scalars and

$$Z_m = \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \sigma_z \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1}$$

where the matrices occurring in the direct product are all 2×2 , and σ_z , which occurs at the m th position, is the third of the three Pauli spinors,

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The overall negative sign in (2.1) ensures that alignment along the positive z direction lowers the energy.

The partition function

$$Q(N, \beta) = \text{Tr} \left(\exp \sum_{m=1}^N \beta \Lambda_m Z_m \right)$$

may be evaluated as a straightforward matrix trace, using the properties of direct

products, as

$$Q(N, \beta) = \prod_{m=1}^N [2 \cosh(\beta \Lambda_m)].$$

The free energy per particle f is given in the thermodynamic limit by

$$-\beta f = \lim_{N \rightarrow \infty} \frac{1}{N} \ln Q(N, \beta) = \frac{1}{2\pi} \int_0^{2\pi} \ln[2 \cosh(\beta \Lambda(\phi))] d\phi$$

where we replace the discrete valued Λ_m by the continuous function $\Lambda(\phi)$, where

$$\Lambda(\phi_m) = \Lambda_m \quad \text{at } \phi_m = 2\pi m/N, \quad m = 1, 2, \dots, N.$$

All quantities of thermodynamic interest may be calculated from f .

In the model we shall be considering, the Hamiltonian can in fact be rotated to the somewhat more general form

$$H = - \sum_{m=1}^N [\frac{1}{2}(1 - \gamma)\Lambda_m^+ + \frac{1}{2}(1 + \gamma)\Lambda_m^-] Z_m \tag{2.2}$$

where Λ_m^\pm are positive scalars and

$$\gamma = Z_1 Z_2 \dots Z_N.$$

The partition function corresponding to this Hamiltonian may be equally readily evaluated,

$$Q(N, \beta) = 2^{N-1} (\prod \cosh(\beta \Lambda_m^+) + \prod \cosh(\beta \Lambda_m^-) + \prod \sinh(\beta \Lambda_m^+) - \prod \sinh(\beta \Lambda_m^-)).$$

The free energy per particle determined from this partition function will depend on the relative magnitudes of Λ_m^+ and Λ_m^- ; for example, when $\Lambda_m^+ > \Lambda_m^-$ the first term will dominate.

3. Algebra of the XY model

We consider a one-dimensional lattice of N sites, labelled $1, 2, \dots, N$. The XY model is given by the following Hamiltonian of nearest-neighbour type:

$$H_1 = - \sum_{m=1}^N (J_1^{xx} X_m X_{m+1} + J_1^{yy} Y_m Y_{m+1}). \tag{3.1}$$

The notation for X_m and Y_m is analogous to that of Z_m in the previous section, so that, for example,

$$[X_m, Y_n] = 2i\delta_{mn} Z_n.$$

We may also include additionally a contribution from an external magnetic field h

$$H_0 = -h \sum_{m=1}^N Z_m. \tag{3.2}$$

The XY model described by $H = H_0 + H_1$ is exactly solvable, and although it does not exhibit a phase transition in the thermodynamic limit for finite β , its thermodynamic behaviour has been extensively studied. Further, the XY model is intimately related to the solution of the two-dimensional Ising model in transfer matrix form (Suzuki

1971); this connection is even more explicit in the case of the generalised model we shall describe in the next section, of which the XY model is only a special case.

We now implement part (a) of our strategy by determining the spectrum generating algebra for the XY model. Define the following matrices γ_r :

$$\left. \begin{aligned} \gamma_r &= Z_1 Z_2 \dots Z_{r-1} X_r \\ \gamma_{N+r} &= Z_1 Z_2 \dots Z_{r-1} Y_r \end{aligned} \right\} \quad r = 2, 3, \dots, N$$

$$\gamma_1 = X_1 \qquad \gamma_{N+1} = Y_1.$$

The matrices $\gamma_r (r = 1, 2, \dots, 2N)$ then generate a Clifford algebra with anticommutation relations given by

$$\{\gamma_r, \gamma_s\} = 2\delta_{rs}.$$

The transformation from $\{X_m, Y_m, Z_m\}$ to $\{\gamma_r\}$ is sometimes called the Jordan–Wigner transformation. Using the γ_r we may construct the $N(2N - 1)$ matrices L_{rs} :

$$L_{rs} = -i/4[\gamma_r, \gamma_s] \quad r, s = 1, 2, \dots, 2N \tag{3.3}$$

which close under the commutation relations of the Lie algebra $SO(2N)$

$$[L_{rs}, L_{pq}] = i(\delta_{rp}L_{sq} - \delta_{sp}L_{rq} + \delta_{rq}L_{ps} - \delta_{sq}L_{pr}). \tag{3.4}$$

(We retain the i in expressions such as (3.3) and (3.4) only when we wish to maintain the Hermiticity of the operators concerned. The algebras, such as $SO(2N)$, that we are interested in are of course real Lie algebras whose defining relations do not involve i .)

From the following expressions, which hold for $m = 1, 2, \dots, N - 1$,

$$X_m X_{m+1} = 2L_{N+m, m+1} \qquad Y_m Y_{m+1} = 2L_{N+m+1, m} \qquad Z_m = 2L_{m, N+m}.$$

We see that in the case of the XY model with *free* ends, where the summation in (3.1) goes from 1 to $N - 1$, we may immediately express H as an element of $SO(2N)$. In the *cyclic* case, however, we require the additional terms $X_N X_1$ and $Y_N Y_1$ and so must enlarge the algebra. This is readily done as follows.

Introduce the matrix $\gamma = Z_1 Z_2 \dots Z_N$. This obeys

$$\{\gamma, \gamma_r\} = 0 \qquad \gamma^2 = 1 \qquad [\gamma, L_{rs}] = 0.$$

Then the operators

$$L_{rs}^{(a)} = \frac{1}{2}(1 - a\gamma)L_{rs} \qquad a = \pm$$

close on the algebra $SO(2N) \oplus SO(2N)$

$$[L_{rs}^{(a)}, L_{pq}^{(b)}] = i\delta_{ab}(\delta_{rs}L_{sq}^{(a)} - \delta_{sp}L_{rq}^{(a)} + \delta_{rq}L_{ps}^{(a)} - \delta_{sq}L_{pr}^{(a)}).$$

This enlarged algebra now contains all the previously required quantities

$$L_{rs} = L_{rs}^+ + L_{rs}^-$$

as well as the cyclic terms

$$X_N X_1 = 2(L_{1, 2N}^{(-)} - L_{1, 2N}^{(+)}),$$

$$Y_N Y_1 = 2(L_{N, N+1}^{(-)} - L_{N, N+1}^{(+)});$$

and so we may write the XY Hamiltonian $H = H_0 + H_1$, equations (3.1) and (3.2), explicitly as an element of $SO(2N) \oplus SO(2N)$. This completes part (a) of our strategy.

4. Translational invariance

The most general element of our $SO(2N) \oplus SO(2N)$ algebra may be written

$$H = \sum_{a=\pm} \sum_{m,n=1}^{2N} \omega_{mn}^{(a)} L_{mn}^{(a)} \tag{4.1}$$

or, more compactly,

$$H = -\text{tr } \omega \mathcal{L}$$

with ω and \mathcal{L} defined as blocked $4N \times 4N$ matrices

$$\omega = \left[\begin{array}{c|c} \omega^{(+)} & 0 \\ \hline 0 & \omega^{(-)} \end{array} \right] \quad \mathcal{L} = \left[\begin{array}{c|c} L^{(+)} & 0 \\ \hline 0 & L^{(-)} \end{array} \right]$$

where the elements of ω^\pm are real numbers and those of \mathcal{L} are the matrices $L_{rs}^{(+)}, L_{rs}^{(-)}$. We now impose translational (more precisely, cyclic) invariance by demanding that H be invariant under the action of the unitary operator \mathcal{I} defined by

$$X_{r+1} = \mathcal{I} X_r \mathcal{I}^{-1} \quad Y_{r+1} = \mathcal{I} Y_r \mathcal{I}^{-1} \quad Z_{r+1} = \mathcal{I} Z_r \mathcal{I}^{-1};$$

that is

$$H = \mathcal{I} H \mathcal{I}^{-1}. \tag{4.2}$$

The operator \mathcal{I} which obeys $\mathcal{I}^N = 1$ and generates a $(2^N \times 2^N)$ -dimensional representation of the cyclic subgroup C_N of $SO(2N) \oplus SO(2N)$, is implemented on \mathcal{L} by

$$\mathcal{I} \mathcal{L}_{RS} \mathcal{I}^{-1} = (\mathcal{D} \mathcal{L} \tilde{\mathcal{D}})_{RS} \quad (\mathcal{D} \tilde{\mathcal{D}} = \mathbf{1})$$

where \mathcal{D} is the $4N \times 4N$ (numerical) matrix defined by

$$\mathcal{D} = \left[\begin{array}{c|c} \mathbf{1} \otimes \Delta^{(+)} & 0 \\ \hline 0 & \mathbf{1} \otimes \Delta^{(-)} \end{array} \right]$$

and tilde denotes matrix transpose.

The cyclic $N \times N$ matrix $\Delta^{(+)}$, and the anticyclic $\Delta^{(-)}$, are given by

$$\Delta^{(+)} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix} \quad \Delta^{(-)} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & \ddots & \\ & & & \ddots & 1 \\ -1 & & & & 0 \end{bmatrix}$$

and obey $\Delta^{(a)} \tilde{\Delta}^{(a)} = \mathbf{1}$, $\Delta^{(a)N} = a \mathbf{1} (a = \pm)$. Imposing the condition (4.2) on the Hamiltonian (4.1) leads to the equation

$$(\mathbf{1} \otimes \Delta^{(a)})^{-1} \omega^{(a)} (\mathbf{1} \otimes \Delta^{(a)}) = \omega^{(a)} \tag{4.3}$$

whose general solution, with $\omega^{(a)}$ antisymmetric, is

$$\omega^{(a)} = \sum_{r=0}^{N-1} (J_r^{(a)} \otimes \Delta^{(a)r} - \tilde{J}_r^{(a)} \otimes \tilde{\Delta}^{(a)r}) \tag{4.4}$$

where the $J_r^{(a)}$ are arbitrary, real 2×2 matrices. This set of coefficients ω in the Hamiltonian (4.1) therefore gives the most general translationally invariant model consistent with the $SO(2N) \oplus SO(2N)$ algebra.

The subscript r in the expression (4.3) for $\omega^{(a)}$ refers to an interaction which is $(\gamma + 1)$ -body and of range r . For example, taking $J_r^+ = J_r^-$, we may rewrite the Hamiltonian (4.1) as

$$H = \sum_{r=0}^{N-1} H_r$$

with

$$H_0 = -h \sum_{m=1}^N Z_m$$

and

$$H_r = - \sum_{m=1}^N (J_r^{xx} X_m Z X_{m+r} + J_r^{yy} Y_m Z Y_{m+r} + J_r^{xy} X_m Z Y_{m+r} + J_r^{yx} Y_m Z X_{m+r})$$

$$r = 1, 2, \dots, N.$$

where we have put

$$J_r^{(a)} = \begin{bmatrix} J_r^{xx} & J_r^{yy} \\ -J_r^{yx} & -J_r^{xy} \end{bmatrix} \quad \text{and} \quad -h = J_0^{xx} + J_0^{yy},$$

and terms like $X_m Z X_{m+r}$ are shorthand for $X_m Z_{m+1} \dots Z_{m+r-1} X_{m+r}$. In this form we see that we may recover the XY model, by choosing all the coefficients except h, J_1^{xx} and J_1^{yy} as zero; as well as other generalisations such as that of Suzuki ($J_r^{xy} = J_r^{yx} = 0$) (Suzuki 1971) and Dzyaloshinsky ($h, J_1^{xx}, J_1^{yy}, J_1^{xy} = -J_1^{yx}$ non-zero) (Siskens *et al* 1974). However, the form (4.1) and (4.3) is most suitable for our purposes, and we now implement part (b) of our general strategy, by choosing a convenient faithful representation of $SO(2N) \oplus SO(2N)$ in which to implement the automorphism (1.1).

5. Diagonalisation of the Hamiltonian

A convenient representation of $SO(2N) \oplus SO(2N)$ in which to implement the automorphism (1.1) is the standard representation of the rotation algebra

$$[S_{rs}, S_{pq}] = \delta_{sp} S_{rq} + \delta_{rq} S_{sp} - \delta_{sq} S_{rp} - \delta_{rp} S_{sq}$$

obtained by setting

$$S_{rs} = e_{rs} - e_{sr} \quad r, s = 1, 2, \dots, 2N$$

where e_{rs} is the $2N \times 2N$ matrix defined by

$$e_{rs}^{mn} = \delta_r^m \delta_s^n \quad m, n = 1, 2, \dots, 2N.$$

We may use this representation for both the '+' and '-' algebras to obtain the

representation for $H^{(a)}$:

$$\hat{H}^{(a)} = (-2i)\omega^{(a)} \quad a = \pm$$

as a $2N \times 2N$ matrix.

Since the $\omega^{(a)}$ are antisymmetric, there exists an automorphism (rotation by an orthogonal matrix) which sends $\omega^{(a)}$ to the canonical form

$$\omega^{(a)} \mapsto \left[\begin{array}{c|c} 0 & -\Lambda^{(a)} \\ \hline \Lambda^{(a)} & 0 \end{array} \right]$$

where the diagonal matrix $\Lambda^{(a)} = \text{diag}\{\Lambda_1^{(a)}, \dots, \Lambda_N^{(a)}\}$ may be chosen to have positive entries which are readily computed (appendix). (This amounts to choosing the commuting elements h_m of the Cartan basis (1.1) proportional to $S_{m,N+m}$ in this representation.)

Since the automorphism

$$\hat{H}^{(a)} \mapsto \sum_{m=1}^N 2i\Lambda_m^{(a)} S_{m,N+m}$$

holds in the $2N$ -dimensional representation, it also holds in the 2^N -dimensional (Hermitian, $S \sim iL^{(a)}$) representation

$$H^{(a)} \mapsto \sum_{m=1}^N -2\Lambda_m^{(a)} L_{m,N+m}^{(a)}$$

so that the original Hamiltonian (4.1) takes the form (2.2)

$$H \mapsto - \sum_{m=1}^N \left[\frac{1}{2}(1 - \gamma)\Lambda_m^+ Z_m + \frac{1}{2}(1 + \gamma)\Lambda_m^- Z_m \right]$$

of the free spin model, and the partition function may be evaluated immediately, completing part (c) of our strategy. The expression for the free energy is given in appendix 1 (equations (A.5) and (A.6)).

6. Conclusion

We have described a generalisation of the spin- $\frac{1}{2}$ XY model which is exactly solvable, and the most general within the context of the $SO(2N) \oplus SO(2N)$ algebra of the usual XY model and translational invariance. The expression derived for the free energy (equations (A.1) and (A.2)) may be chosen to have particularly simple, closed forms; for in the infinite-range limit the coupling constants in (A.2) can be taken as the Fourier coefficients of (fairly arbitrary) functions.

The Hamiltonian considered, though of doubtful direct physical interest, has a useful interpretation as the associated Hamiltonian of the two-dimensional Ising problem; that is, an operator commuting with the transfer matrix. We show this in appendix 2, where we also derive the Onsager solution. Although we have only considered translationally invariant models in this paper it is a straightforward matter to proceed to the non-invariant case. For example, if the Hamiltonian is invariant under ν -translations, where ν is some positive integer dividing N ,

$$\mathcal{I}^\nu H \mathcal{I}^{-\nu} = H$$

we obtain the general solution by solving the modification of (4.3)

$$(\mathbf{1} \otimes \Delta^{(a)})^{-\nu} \omega^{(a)} (\mathbf{1} \otimes \Delta^{(a)})^{\nu} = \omega^{(a)}.$$

Appendix 1

We determine the elements $\Lambda_m^{(a)}$ of the canonical form of

$$\omega^{(a)} = \sum_{r=0}^{N-1} (J_r^{(a)} \otimes \Delta^{(a)r} - \tilde{J}_r \otimes \tilde{\Delta}^{(a)r}). \tag{A.1}$$

Assume that the eigenvalue equation for $\omega^{(a)}$ is of the form

$$\omega^{(a)}(\mathbf{v} \otimes \mathbf{e}^{(a)}) = i\mu(\mathbf{v} \otimes \mathbf{e}^{(a)}) \tag{A.2}$$

where $\mathbf{e}^{(a)}$ is an eigenvector of $\Delta^{(a)}$ with eigenvalue $\lambda^{(a)}$

$$\Delta^{(a)}\mathbf{e}^{(a)} = \lambda^{(a)}\mathbf{e}^{(a)}.$$

By substituting the expression (A.1) for $\omega^{(a)}$ in (A.2) we obtain

$$\sum_{r=0}^{N-1} (J_r^{(a)} \lambda^{(a)r} - \tilde{J}_r \lambda^{(a)-r}) \mathbf{v} \otimes \mathbf{e}^{(a)} = i\mu \mathbf{v} \otimes \mathbf{e}^{(a)}$$

so that μ is the eigenvalue corresponding to the eigenvector \mathbf{v} of the 2×2 Hermitian matrix $M^{(a)}$:

$$M^{(a)} = -i \sum_{r=0}^{N-1} (J_r^{(a)} \lambda^{(a)r} - \tilde{J}_r \lambda^{(a)-r}) \tag{A.3}$$

Rewriting $M^{(a)}$ in terms of the four real numbers $m_{\mu}^{(a)}$

$$M^{(a)} = \sum_{\mu=0}^3 m_{\mu}^{(a)} \sigma_{\mu} \tag{A.4}$$

the two eigenvalues of $M^{(a)}$ are immediately given by

$$\mu = m_0^{(a)} + (m_1^{(a)2} + m_2^{(a)2} + m_3^{(a)2})^{1/2} \tag{A.5}$$

and

$$\mu = m_0^{(a)} - (m_1^{(a)2} + m_2^{(a)2} + m_3^{(a)2})^{1/2}.$$

Since the eigenvalues of $\omega^{(a)}$ occur in conjugate pairs, the N positive values $\Lambda_m^{(a)}$ are enumerated by taking the modulus of (A.5) corresponding to each eigenvalue $\lambda_m^{(a)}$ of $\Delta^{(a)}$:

$$\lambda_m^{(a)} = \exp(i\phi_m^{(a)}) \quad m = 1, 2, \dots, N$$

with

$$\phi_m^{(+)} = 2m\pi/N, \quad \phi_m^{(-)} = (2m + 1)\pi/N.$$

Defining the energy function $\Lambda^{(a)}(\phi)$ by

$$\Lambda^{(a)}(\phi_m) = \Lambda_m^{(a)}$$

we have explicitly

$$\Lambda^{(a)}(\phi) = |m_0^{(a)}(\phi) + (m_1^{(a)}(\phi)^2 + m_2^{(a)}(\phi)^2 + m_3^{(a)}(\phi)^2)^{1/2}|$$

with the functions $m_\mu^{(a)}(\phi)$ given in terms of the matrix elements of $J_r^{(a)}$ by

$$\begin{aligned}
 m_0^{(a)}(\phi) &= \sum_{r=1}^{N-1} (J_r^{(a)11} + J_r^{(a)22}) \sin(r\phi) \\
 m_1^{(a)}(\phi) &= \sum_{r=1}^{N-1} (J_r^{(a)12} + J_r^{(a)21}) \sin(r\phi) \\
 m_2^{(a)}(\phi) &= \sum_{r=0}^{N-1} (J_r^{(a)12} - J_r^{(a)21}) \cos(r\phi) \\
 m_3^{(a)}(\phi) &= \sum_{r=1}^{N-1} (J_r^{(a)11} - J_r^{(a)22}) \sin(r\phi).
 \end{aligned}$$

The magnetic field term occurs explicitly as the $r = 0$ component of $J_2^{(a)}(\phi)$:

$$h = -\frac{1}{2}(m_2^{(+)}(0) + m_2^{(-)}(0)).$$

In the case $J_r^+ = J_r^-$ we have $\Lambda_m^+(\phi) = \Lambda_m^-(\phi)$ as $N \rightarrow \infty$, and so the free energy may be written

$$-\beta f = \frac{1}{2\pi} \int_0^{2\pi} \ln[2 \cosh(\beta \Lambda(\phi))] d\phi \tag{A.6}$$

as in § 2, with

$$\begin{aligned}
 \Lambda(\phi) &= \sum_{r=1}^{\infty} a_r \sin(r\phi) \\
 &+ \left[\left(\sum_{r=1}^{\infty} b_r \sin(r\phi) \right)^2 + \left(\sum_{r=1}^{\infty} c_r \cos(r\phi) - h \right)^2 + \left(\sum_{r=1}^{\infty} d_r \sin(r\phi) \right)^2 \right]^{1/2} \tag{A.7}
 \end{aligned}$$

writing

$$\begin{aligned}
 a_r &= J_r^{yx} - J_r^{xy} & b_r &= J_r^{yy} - J_r^{xx} \\
 c_r &= J_r^{xx} + J_r^{yy} & d_r &= J_r^{xy} + J_r^{yx}.
 \end{aligned}$$

All the thermodynamic quantities may be calculated from (A.6) in the usual way.

Appendix 2

We exhibit the relation of the generalised XY model to the two-dimensional Ising model, and deduce the Suzuki equivalence result (Suzuki 1971). We follow Kasteleyn's treatment of the Ising problem below (Kasteleyn 1975).

The Hamiltonian for a two-dimensional Ising model on a $M \times N$ lattice may be written

$$H = - \sum_{j=1}^N \sum_{k=1}^M (J_1 \sigma_{j,k}^z \sigma_{j+1,k}^z + J_2 \sigma_{j,k}^z \sigma_{j,k+1}^z) \tag{A.8}$$

where the Pauli spinor $\sigma_{j,k}^z$ is that associated with a spin having the values +1 or -1 at the (j, k) site. The partition function

$$Q = \text{Tr}[\exp(-\beta H)]$$

is a classical sum over all 2^{MN} allowed configurations, and is conventionally evaluated by introducing the transfer matrix

$$T(\mu^{(k)}, \mu^{(k+1)}) = \exp(-E_L(\mu^{(k)}, \mu^{(k+1)})).$$

Here one denotes the state of row k by

$$\mu^{(k)} = (\mu_{1k}, \mu_{2k}, \dots, \mu_{Nk})$$

and the 'layer' energy $E_L(\mu^{(k)}, \mu^{(k+1)})$ gives the contribution to the total energy from the mutual interactions in row k , and the interaction between rows k and $k+1$. With this notation the partition function Q may be rewritten as

$$Q = \sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \dots \sum_{\mu^{(N)}} T(\mu^{(1)}, \mu^{(2)}) T(\mu^{(2)}, \mu^{(3)}) \dots T(\mu^{(N)}, \mu^{(1)})$$

(imposing cyclic boundary conditions so that row M interacts with row 1) or, more elegantly,

$$Q = \text{Tr } T^M$$

where we consider $T(\mu^{(k)}, \mu^{(k+1)})$ as the $(\mu^{(k)}, \mu^{(k+1)})$ element of a $2^N \times 2^N$ matrix T , the transfer matrix.

For the Ising Hamiltonian (A.8) the layer energy may be written

$$E_2(\mu, \mu') = - \sum_{j=1}^N (J_1 \mu_j \mu_{j+1} + J_2 \mu_j \mu'_j)$$

and the transfer matrix elements become

$$T(\mu, \mu') = T_2(\mu, \mu) T_1(\mu, \mu') \quad T_1(\mu, \mu') = \exp\left(\sum_{j=1}^N K_2 \mu_j \mu'_j\right)$$

$$T_2(\mu, \mu) = \exp\left(\sum_{j=1}^N K_1 \mu_j \mu_{j+1}\right)$$

where we have written $K_1 = \beta J_1$, $K_2 = \beta J_2$.

In our notation we may write

$$T_1 = \prod_{j=1}^N (e^{K_2} + e^{-K_2} X_j) \quad T_2 = \exp\left(\sum_{j=1}^N K_1 Z_j Z_{j+1}\right)$$

where we have extended the definition of T_2 to

$$T_2(\mu, \mu') = \exp\left(\sum_{j=1}^N K_1 \mu_j \mu_{j+1}\right) \prod_j \delta(\mu_j, \mu'_j).$$

We may thus rewrite the transfer matrix T as

$$T = T_2 T_1 = C^N \exp\left(K_1 \sum_{j=1}^N Z_j Z_{j+1}\right) \exp\left(K_2^* \sum_{j=1}^N X_j\right)$$

re-expressing T_1 as an exponential, with the identification

$$C \cosh K_2^* = e^{K_2} \quad C \sinh K_2^* = e^{-K_2}.$$

We may reorient the (X_i, Y_i, Z_i) axes—by a $\Pi_i \text{SO}(3)_i$ rotation—and so rewrite the

transfer matrix in its final form

$$T = C^N \exp\left(K_1 \sum_{j=1}^N X_j X_{j+1}\right) \exp\left(K_2^* \sum_{j=1}^N Z_j\right).$$

as given, for example, by Kasteleyn.

Noting that T is a product of exponents of elements of the Lie algebra treated in this paper, we infer that it is an element of the associated Lie group. We may diagonalise T by using the representation of § 5 for which we have essentially

$$\sum_{j=1}^N X_j X_{j+1} \sim 2i \begin{bmatrix} & \lambda^{-1} \\ -\lambda & \end{bmatrix} \quad \sum_{j=1}^N Z_j \sim 2i \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

(In this rather succinct notation, suggested by appendix 1, the terms on the right must be read as *pairs* of $2N \times 2N$ matrices; we have suppressed the pairs as each element has the same form. Further, the symbol λ stands for the $\Delta^{(a)}$ matrix of § 4. Alternatively, λ may be treated as an indeterminate to be replaced by the eigenvalues of Δ^+ and Δ^- in the final calculation. In both cases we shall suppress the superscript $(a) = \pm$.)

We therefore have in the representation

$$\hat{T} = C^N (\hat{T}^{(+)} + \hat{T}^{(-)})$$

where

$$\begin{aligned} \hat{T}^{(a)} &= \exp\left(2iK_1 \begin{bmatrix} & \lambda^{-1} \\ -\lambda & \end{bmatrix}\right) \exp\left(2iK_2^* \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\right) \\ &= \begin{bmatrix} C_1 C_2 - \lambda^{-1} S_1 S_2 & -i(C_1 S_2 - \lambda^{-1} S_1 C_2) \\ i(C_1 S_2 - \lambda S_1 C_2) & C_1 C_2 - \lambda S_1 S_2 \end{bmatrix} \end{aligned} \tag{A.9}$$

putting

$$\begin{aligned} C_1 &= \cosh(2K_1) & C_2 &= \cosh(2K_2^*) \\ S_1 &= \sinh(2K_1), & S_2 &= \sinh(2K_2^*). \end{aligned}$$

Note that we may rewrite $\hat{T}^{(a)}$ in the form of (A.4):

$$\hat{T}^{(a)} = \sum_{\mu=0}^3 t_{\mu}^{(a)} \sigma_{\mu}$$

with

$$\begin{aligned} t_0^{(a)} &= C_1 C_2 - S_1 S_2 \cos q, & t_1^{(a)} &= -S_1 C_2 \sin q, \\ t_2^{(a)} &= C_1 S_2 - S_1 C_2 \cos q, & t_3^{(a)} &= i S_1 S_2 \sin q \end{aligned}$$

where $\lambda \equiv \exp(iq)$ (again suppressing the supercript (a) which λ and q should carry).

In this form we see that the transfer matrix is just a (complex) generalised XY operator. We may recover a real XY Hamiltonian by an imaginary rotation of $-2iK_2^*$ about the $\sigma_y (= \sigma_2)$ axis, so that

$$\hat{T}^{(a)} \mapsto \sum_{\mu=0}^3 t'_{\mu}^{(a)} \sigma_{\mu}$$

$$t'_{\mu} = (C_1 C_2 - S_1 S_2 \cos q, -S_1 \sin q, C_1 S_2 - S_1 C_2 \cos q, 0).$$

This rotated transfer matrix commutes with the conventional XY Hamiltonian

$$H_{XY} = H_0 + H_1$$

for which

$$\hat{H}_{XY}^{(a)} = 2m_\mu^{(a)}\sigma_\mu$$

$$m_\mu^{(a)} = (0, \sin[q(J_1^{XX} - J_1^{YY})], \cos[q(J_1^{XX} + J_1^{YY})] - h, 0)$$

if the corresponding three-vectors $(t_1^{(a)}, t_2^{(a)}, t_3^{(a)})$ and $(m_1^{(a)}, m_2^{(a)}, m_3^{(a)})$ are parallel; that is, if $J_1^{XX} : J_1^{YY} : h = \exp(2K_2) : \exp(-2K_2) : 2 \coth(2K_1)$ which is the Suzuki equivalence result.

We may also readily diagonalise the transfer matrix T in this representation. Diagonalisation corresponds to a rotation of T :

$$T \mapsto C^N \exp\left(\sum_{i=1}^N \mu_i Z_i\right). \quad (\text{A.10})$$

In our representation this corresponds to

$$\hat{T}^{(a)} \mapsto \exp\left(2i \begin{bmatrix} & -\mu \\ \mu & \end{bmatrix}\right)$$

where μ is a diagonal $N \times N$ matrix, $\mu = \text{diag}(\mu_1, \mu_2, \dots, \mu_N)$ and there is actually a suppressed index (a) corresponding to the pair of matrices on the right. Equating traces, using the explicit expression of $\hat{T}^{(a)}$ given in (A.9), we obtain

$$\cosh(2\mu) = C_1 C_2 - S_1 S_2 \cos q$$

for the eigenvalues μ_i . Returning to the original $2^N \times 2^N$ formulation of (A.10) we see that the maximum eigenvalue of T —the one of physical interest—is given by $C^N \prod_{i=1}^N \exp \mu_i$, where

$$\cosh(2\mu_i) = \cosh(2K_1) \cosh(2K_2^*) - \sinh(2K_1) \sinh(2K_2) \cos(2\pi i/N),$$

Onsager's classic result.

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